One problem involving Pell equation
(attached to problem
https://www.linkedin.com/feed/update/urn:li:activity:6512525019552321536)
Proposed by Arkady Alt.
Find all natural m such that $2m+1$ and $3m+1$ both are perfect squares;
Solution.
a) $2m+1 = x^2, 3m+1 = y^2$ then $3x^2 - 2y^2 = 1$, where $x \equiv 1 (\text{mod}\, 2), y \equiv 1 (\text{mod}\, 3)$.
Let $P_+ := \left\{(x,y) \mid x,y \in \mathbb{N} \text{ and } 3x^2 - 2y^2 = 1\right\}$,
$\overline{P}_+ := \left\{\left(x\sqrt{3} + y\sqrt{2}\right) \mid x,y \in \mathbb{N} \text{ and } 3x^2 - 2y^2 = 1\right\}$ and $\varphi : P_+ \to \overline{P}_+$
defined by $\varphi(x,y) = x\sqrt{3} + y\sqrt{2}$. Since $x\sqrt{3} + y\sqrt{2} = 0 \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$ then
φ isomorphism.
Let x_n, y_n be natural numbers defined by representation $\left(\sqrt{3} + \sqrt{2}\right)\left(5 + 2\sqrt{6}\right)^n$
in the form $x_n\sqrt{3} + y_n\sqrt{2}$ obtained by collected terms with $\sqrt{3}$ and $\sqrt{2}$.
Since $3x_n^2 - 2y_n^2 = \left(\sqrt{3} + \sqrt{2}\right)\left(5 + 2\sqrt{6}\right)^n\left(\sqrt{3} - \sqrt{2}\right)\left(5 - 2\sqrt{6}\right)^n = 1$ then
$(x_n, y_n) \in P_+$ for any $n \in \mathbb{N} \cup \{0\}$ and can be defined recursively by
$x_{n+1}\sqrt{3} + y_{n+1}\sqrt{2} = \left(x_n\sqrt{3} + y_n\sqrt{2}\right)\left(5 + 2\sqrt{6}\right) \Leftrightarrow$

(1) $\begin{cases} x_{n+1} = 5x_n + 4y_n \\ y_{n+1} = 6x_n + 5y_n \end{cases}, n \in \mathbb{N} \cup \{0\}$,

where $x_0 = 1, y_0 = 1$.
We will prove that for any solution $(x,y) \in P_+$ there is $n \in \mathbb{N} \cup \{0\}$ such that
$(x,y) = (x_n, y_n)$, that is $x\sqrt{3} + y\sqrt{2} = x_n\sqrt{3} + y_n\sqrt{2}$.
Proof.
Suppose opposite, that is $x\sqrt{3} + y\sqrt{2} \neq x_n\sqrt{3} + y_n\sqrt{2}$ for any $n \in \mathbb{N} \cup \{0\}$.
Since $x_n\sqrt{3} + y_n\sqrt{2} < x_{n+1}\sqrt{3} + y_{n+1}\sqrt{2}$ for any $n \in \mathbb{N} \cup \{0\}$ and
$\lim\limits_{n\to\infty}\left(x_n\sqrt{3} + y_n\sqrt{2}\right) = \infty$ then there is $m \in \mathbb{N} \cup \{0\}$ such that
$x_m\sqrt{3} + y_m\sqrt{2} < x\sqrt{3} + y\sqrt{2} < x_{m+1}\sqrt{3} + y_{m+1}\sqrt{2}$.
Since $\left(x\sqrt{3} + y\sqrt{2}\right)\left(5 - 2\sqrt{6}\right) = (5x - 4y)\sqrt{3} + (5y - 6x)\sqrt{2}$ and
$5x > 4y \Leftrightarrow 25x^2 > 16y^2 \Leftrightarrow 75x^2 > 48y^2 \Leftrightarrow 25(2y^2 + 1) > 48y^2 \Leftrightarrow 2y^2 + 25 > 0$,
$5y > 6x \Leftrightarrow 25y^2 > 36x^2 \Leftrightarrow 50y^2 > 72x^2 \Leftrightarrow 25(3x^2 - 1) > 72x^2 \Leftrightarrow 3x^2 > 25$
(because x is odd and $x \neq 1$ then $x \geq 3$ and latter inequality holds).
Also note that no $(x,y) \in P_+$ such that
$x\sqrt{3} + y\sqrt{2} \in \left(x_0\sqrt{3} + y_0\sqrt{2}, x_1\sqrt{3} + y_1\sqrt{2}\right) = \left(\sqrt{3} + \sqrt{2}, 9\sqrt{3} + 11\sqrt{2}\right)$.
Indeed, since x is odd and $\dfrac{3x^2 - 1}{2}$ isn't perfect square for $x \in \{3,5,7\}$ and $x \notin \{1,9\}$
(because otherwise $y \in \{1,11\}$ and $x\sqrt{3} + y\sqrt{2} \in \left\{\sqrt{3} + \sqrt{2}, 9\sqrt{3} + 11\sqrt{2}\right\}$) then
$x > 9 \Rightarrow 2y^2 = 3x^2 - 1 > 242 \Rightarrow y > 11$ and, therefore $m \geq 1$.
So, if $x\sqrt{3} + y\sqrt{2} \in \left(x_m\sqrt{3} + y_m\sqrt{2}, x_{m+1}\sqrt{3} + y_{m+1}\sqrt{2}\right)$ then
$\left(x\sqrt{3} + y\sqrt{2}\right)\left(5 - 2\sqrt{6}\right) = (5x - 4y)\sqrt{3} + (5y - 6x)\sqrt{2} \in \left(x_{m-1}\sqrt{3} + y_{m-1}\sqrt{2}, x_m\sqrt{3} + y_m\sqrt{2}\right.$
and $(5x - 4y, 5y - 6x) \in P_+$.

Applying reverse transformation m times, that is multiplying inequality
$x_m\sqrt{3} + y_m\sqrt{2} < x\sqrt{3} + y\sqrt{2} < x_{m+1}\sqrt{3} + y_{m+1}\sqrt{2}$ by $\left(5-2\sqrt{6}\right)^m$
we obtain inequality $x_0\sqrt{3} + y_0\sqrt{2} < x'\sqrt{3} + y'\sqrt{2} < x_1\sqrt{3} + y_1\sqrt{2}$
where $x'\sqrt{3} + y'\sqrt{2} = \left(x\sqrt{3} + y\sqrt{2}\right)\left(5-2\sqrt{6}\right)^m$ and $x', y' = x', y'$ are natural.
But as was proved above $\overline{P}_+ \cap \left(\sqrt{3} + \sqrt{2}, 9\sqrt{3} + 11\sqrt{2}\right) = \varnothing$.
Obtained contradiction complete the proof.
Thus, $P_+ = \{(x_n, y_n) \mid n \in \mathbb{N} \cup \{0\}\}$.
Instead system (**1**) convenient to use independent recurrences of the second degree defined x_n, y_n. Namely,

$$(1) \Leftrightarrow \begin{cases} x_{n+2} - 10x_{n+1} + x_n = 0, n \in \mathbb{N} \cup \{0\}, x_0 = 1, x_1 = 9 \\ y_{n+2} - 10y_{n+1} + y_n = 0, n \in \mathbb{N} \cup \{0\}, y_0 = 1, y_1 = 11 \end{cases}.$$